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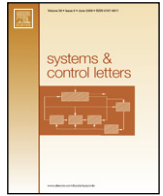
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Extension of Kalman–Yakubovich–Popov lemma to descriptor systems

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ABSTRACT

This paper presents a complete analogue of the well-known Kalman–Yakubovich–Popov lemma for descriptor systems, i.e. necessary and sufficient linear matrix inequality conditions for passivity and positive realness of descriptor systems. Also a full characterization of extended strictly positive realness is given for this class of systems. Some of the earlier related results are recovered from the presented results.

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1. Introduction

The notion of passivity has always been of interest in various problems of systems and control theory. It is intimately related to the notion of positive realness. The relation between these two properties has been under investigation ever since Kalman's introduction of state space approach. The very well-known Kalman–Yakubovich–Popov (KYP) lemma is among the classical results of systems theory. For more than four decades, many researchers have investigated passivity/positive realness and their various extensions within the framework of state space systems. As an encyclopedic account of this vast literature, we refer to [1].

One line of research consists of efforts in extending the available literature for state space systems to descriptor systems. Despite the considerable contributions of numerous papers, a full analogue of KYP lemma for descriptor systems has not appeared yet to the best of author's knowledge. The very aim of this paper is to fill this gap by providing the extension of KYP lemma to descriptor systems. To do so, we first formulate passivity in terms of the so-called dissipation inequality by following Jan Willems' conceptual framework that is introduced in the seminal paper [2]. This will be followed by

necessary and sufficient linear matrix inequality (LMI)-type conditions for passivity. Our treatment is highly inspired by the approach of [3]. In particular, the refinement of the Weierstrass form for realizations of positive real transfer matrices that was proven in [3] serves as one of the key tools in our development.

Vast majority of the related research is concentrated on strict versions of positive realness and/or works under extra assumptions. In [4,5], the authors focus on impulse-free descriptor systems and provide LMI-type conditions for passivity and/or positive realness. As we discuss in Remark 4.3, checking passivity of an impulse-free descriptor system is equivalent to checking passivity of a corresponding standard state space system. In the current paper, we do not impose impulse-freeness on the systems we deal with. As such, the results of [4,5] can be obtained as special cases from the main results of our paper. In [3], the authors present two separate necessary LMI conditions for positive realness. One of these become also sufficient with an extra assumption on the feed-through term. All these results can be recovered from necessary and sufficient conditions presented in this paper (see Section 6).

The KYP lemma has also been studied in the behavioral framework (see [6,7]). In [6], the authors present necessary and sufficient conditions for dissipativity of generalized first-order systems. The paper [7] provides conditions for both first- and higher-order systems. All these results are obtained in the more general framework of behavioral systems. However, they do not yield explicit LMI conditions in terms of system matrices of a state space system as it is done in this paper.

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The organization of the paper is as follows. After introducing the notational conventions in Section 2, we review descriptor systems, their minimality, Weierstrass form, definition and certain properties of solution in Section 3. This will be followed by Section 4 where the notion of passivity, positive realness, and its variations of interest are reviewed. Section 5 contains the main results and their proofs while Section 6 is devoted to recover some earlier results as special cases of the main results of this paper. The paper ends with the conclusions in Section 7 and review of the some known results and their variations that we employed in the current paper in Appendix.

2. Notation

The following notations and conventions will be in force. The symbols \mathbb{R} , \mathbb{R}_+ , \mathbb{C} and \mathbb{C}_+ denote the sets of real numbers, non-negative real numbers, complex numbers and complex numbers with positive real part, respectively. The notation $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices with real elements and $\mathbb{R}^{n \times m}(s)$ the set of $n \times m$ matrices of rational functions. For a complex number s , $\text{Re}(s)$ stands for the real part. For a complex vector v , the conjugate, the transpose and the conjugate transpose are denoted, respectively, \bar{v} , v^T and v^H . These conventions are used for matrices in the obvious manner. Let M be a matrix. The image of M is denoted by $\text{im } M$ and kernel of M by $\ker M$. Let P be a square matrix. The matrix P is said to be *Hurwitz* if all eigenvalues of P have strictly negative real part. The matrix P is said to be *symmetric* if $P = P^T$. We say that P (not necessarily symmetric) is *positive semi-definite* if $v^T P v \geq 0$ for all vectors v . It is said to be *positive definite* if it is positive semi-definite and $v^T P v = 0$ implies $v = 0$. We write $P \geq 0$ and $P > 0$ by meaning that P is positive semi-definite and positive definite, respectively. Negative (semi-)definiteness is defined in a similar fashion. The notation

$$M \stackrel{w}{\geq} 0 \quad \text{and} \quad M \stackrel{w}{>} 0$$

stands for $w^T M w \geq 0$ and $w^T M w > 0$ for all $w \in \mathcal{W}$. Given a matrix $A \in \mathbb{R}^{n \times m}$, $\text{Sym}(A)$ stands for the matrix $A + A^T$. Given two vectors u and v , the notation $\text{col}(u, v)$ denotes the vector obtained by stacking u over v . The identity matrix will be denoted by I , while the zero matrix by 0 . A rational matrix $G(s)$ is said to be *proper* if $\lim_{s \rightarrow \infty} G(s)$ is finite.

3. Preliminaries

In what follows, we introduce/review some of the concepts that will be used later.

3.1. Descriptor systems

Consider the descriptor system

$$E \dot{x}(t) = Ax(t) + Bu(t) \quad (1a)$$

$$y(t) = Cx(t) + Du(t) \quad (1b)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and the matrices (A, B, C, D, E) are of appropriate sizes. We denote (1) by $\Sigma(E, A, B, C, D)$.

We assume that (1) is regular, i.e. $sE - A$ is invertible as a polynomial matrix.

Throughout the paper, we are interested in particular type of solutions. Let $L_{2,\text{loc}}(\mathbb{R}, \mathbb{R}^*)$, $AC(\mathbb{R}, \mathbb{R}^*)$, and $C^\infty(\mathbb{R}, \mathbb{R}^*)$ denote, respectively, the set of all locally integrable, absolutely continuous, smooth functions defined from \mathbb{R} to \mathbb{R}^* . We say that

- $(x, u) \in AC(\mathbb{R}, \mathbb{R}^n) \times L_{2,\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ is a *solution* if (1a) is satisfied for almost all $t \in \mathbb{R}$.

- $(x, u) \in C^\infty(\mathbb{R}, \mathbb{R}^{n+m})$ is a *smooth solution* if (1a) is satisfied for all $t \in \mathbb{R}$.

Let \mathcal{B} and \mathcal{B}_s be the set of all solutions and smooth solutions, respectively.

The transfer matrix associated with (1) is given by $D + C(sE - A)^{-1}B$. We say that a descriptor system is *minimal* if there is no other descriptor system with less number of states yielding the same transfer matrix.

We quote the following well-known theorem that states necessary and sufficient conditions for minimality.

Theorem 3.1 ([8,9]). *Let*

$$G(s) = C(sE - A)^{-1}B + D \quad (2)$$

be a rational function where E and A are the square matrices with dimension n . Then, (1) is a minimal realization of $G(s)$ if, and only if, the following conditions are satisfied:

- $\text{rank} \begin{bmatrix} A - sE & B \end{bmatrix} = n$ for all $s \in \mathbb{C}$ (*Finite controllability*).
- $\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$ (*Infinite controllability*).
- $\text{rank} \begin{bmatrix} A^T - sE^T & C^T \end{bmatrix} = n$ for all $s \in \mathbb{C}$ (*Finite observability*).
- $\text{rank} \begin{bmatrix} E^T & C^T \end{bmatrix} = n$ for all $s \in \mathbb{C}$ (*Infinite observability*).
- $A \ker E \subseteq \text{im } E$ (*Absence of non-dynamics modes*).

3.2. Weierstrass form

A useful tool in the analysis of descriptor systems is the Weierstrass form. If (1) is regular, there exist two square invertible matrices S and T such that the system (1) is transformed to the Weierstrass canonical form

$$\tilde{E} \dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) \quad (3a)$$

$$y(t) = \tilde{C}x(t) + Du(t) \quad (3b)$$

with

$$\tilde{E} = SET = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{A} = SAT = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix},$$

$$\tilde{B} = SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \tilde{C} = CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent, i.e. $N^q = 0$ for some integer $q \geq 0$. We denote the smallest of such integers by k .

3.3. Properties of solutions

The following lemma deals with certain properties of solutions that will be used later.

Lemma 3.2. *For the descriptor system (1), there exist matrices $V \in \mathbb{R}^{n_1 \times \ell}$, $F \in \mathbb{R}^{\ell \times \ell}$, $U \in \mathbb{R}^{m \times \ell}$ and $W \in \mathbb{R}^{\ell \times (2n_1 + m)}$ with $\ell = n_1 + (k+1)m$ with $EVF = AV + BU$ such that the following statements hold.*

- (1) If $(x, u) \in \mathcal{B}$ is a solution then $\text{col}(\dot{x}(t), x(t), u(t)) \in \text{im col}(VF, V, U)$ for almost all $t \in \mathbb{R}$.
- (2) If $\zeta \in \text{im col}(VF, V, U)$ then there exists a smooth solution $(x, u) \in \mathcal{B}_s$ such that

$$\text{col}(\dot{x}(t), x(t), u(t)) = \zeta. \quad (4)$$

- (3) If $(x, u) \in \mathcal{B}$, where u is a constant function, is a solution then

$$\text{col}(\dot{x}(t), x(t), u(t)) \in \text{im}(\text{col}(VF, V, U)W)$$

for almost all $t \in \mathbb{R}$.

- (4) If $\zeta \in \text{im}(\text{col}(VF, V, U)W)$ then there exists a smooth solution (x, u) , where u is a constant function, such that $\text{col}(\dot{x}(0), x(0), u(0)) = \zeta$.

Proof. 1: Without loss of generality, we can assume that (E, A, B, C) is in the Weierstrass form. Then, any trajectory of (1) is given by (see for instance [10])

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) \quad (5)$$

$$x_2(t) = - \sum_{i=0}^{k-1} \frac{d^i}{dt^i} (N^i B_2 u(t)) \quad (6)$$

for almost all $t \in \mathbb{R}$. By differentiating the second equation, we get

$$\dot{x}_2(t) = - \sum_{i=0}^{k-1} \frac{d^{(i+1)}}{dt^{(i+1)}} (N^i B_2 u(t)) \quad (7)$$

for almost all $t \in \mathbb{R}$. By putting (5) and (7) together, we get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ x_1(t) \\ x_2(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -B_2 & \cdots & -N^{k-2}B_2 & -N^{k-1}B_2 \\ \hline I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -B_2 & -NB_2 & \cdots & -N^{k-1}B_2 & 0 \\ \hline 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ u(t) \\ v_1(t) \\ \vdots \\ v_{k-1}(t) \\ v_k(t) \end{bmatrix} \quad (8)$$

for almost all $t \in \mathbb{R}$ where $v_i(t)$ are functions satisfying $N^i B_2 v_i(t) = \frac{d^i}{dt^i} (N^i B_2 u(t))$ for $i = 0, 1, \dots, k$. Then, the choices

$$V = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -B_2 & -NB_2 & \cdots & -N^{k-1}B_2 & 0 \end{bmatrix} \quad (9)$$

$$F = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \xi(t) = \begin{bmatrix} x_1(t) \\ u(t) \\ v_1(t) \\ \vdots \\ v_{k-1}(t) \\ v_k(t) \end{bmatrix} \quad (10)$$

$$U = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (11)$$

prove the claim.

2: Let

$$\zeta = \begin{bmatrix} VF \\ V \\ U \end{bmatrix} \text{col}(\xi, \eta_0, \eta_1, \dots, \eta_k)$$

where $\xi \in \mathbb{R}^{n_1}$ and $\eta_i \in \mathbb{R}^m$ for $i \in \{0, 1, \dots, k\}$. Define

$$u(t) = \sum_{i=0}^k \frac{\eta_i}{i!} t^i$$

and

$$v_i(t) = \frac{d^i u}{dt^i}(t).$$

Let x_1 be the unique solution of the differential equation

$$\dot{x}_1 = A_1 x_1 + B_1 u$$

with $x_1(0) = \xi$. Take

$$\begin{bmatrix} \dot{x}(0) \\ x(0) \\ u(0) \end{bmatrix} = \begin{bmatrix} VF \\ V \\ U \end{bmatrix} \begin{bmatrix} x_1(t) \\ u(t) \\ v_1(t) \\ \vdots \\ v_{k-1}(t) \\ v_k(t) \end{bmatrix}.$$

It follows from (8) that (x, u) is a smooth solution. Moreover, (4) is satisfied by construction.

3 and 4: By taking

$$W = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (12)$$

that conforms the partitions given in (8)–(11) and by repeating the arguments given above. ■

We define

$$\mathcal{W} = \text{im} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} \quad \text{and} \quad \overline{\mathcal{W}} = \text{im} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} W. \quad (13)$$

4. Passivity and positive realness

Following Willems [2,11], we formulate the notion of passivity via the so-called dissipation inequality.

Definition 4.1. The system (1) is *passive* if there exists a non-negative-valued function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t) dt \geq V(x(t_1))$$

for all t_0, t_1 with $t_1 \geq t_0$, $(x, u) \in \mathcal{B}$, and $y = Cx + Du$. If exists, V is called a *storage function*.

An intimately related concept is positive realness.

Definition 4.2. A rational matrix $G(s) \in \mathbb{R}^{m \times m}(s)$ is

- (1) *positive real (PR)* if
 - G is analytic in \mathbb{C}_+ ,
 - $G(s) + G^H(s) \geq 0$ for all $s \in \mathbb{C}_+$;
- (2) *strictly positive real (SPR)* if
 - G is PR,
 - $i\omega$ is not a pole of $G(s)$,
 - $G(i\omega) + G^H(i\omega) > 0$ for all $\omega \in [0, \infty)$;
- (3) *extended strictly positive real (ESPR)* if
 - $G(s)$ is SPR,
 - $G(i\infty) + G^H(i\infty) > 0$.

Remark 4.3. A commonly used assumption in the study of positive realness of descriptor systems (see for instance [4,5]) is *impulsive freeness* (i.e. $N = 0$ in (3)). Note that

$$G(s) = C_1(sI - A_1)^{-1} + D - C_2 B_2$$

when (E, A, B, C) is given in the Weierstrass form (3) and $N = 0$. Obviously, the system (E, A, B, C, D) yields an (E)(S)PR transfer matrix if and only if $(I, A_1, B_1, C_1, D - C_2 B_2)$ yields an (E)(S)PR transfer matrix. This means that the assumption $N = 0$ brings one to the standard state space framework.

Remark 4.4. In a number of papers (see [4] for continuous time and [12] for discrete time), a rational matrix $G(s) \in \mathbb{R}^{m \times m}(s)$ is defined as SPR if

- G is analytic in \mathbb{C}_+ ;
- $G(i\omega) + G^H(i\omega) > 0$ for all $\omega \in [0, \infty)$.

When $E = I$ and $N = 0$, these two conditions guarantee that $G(s)$ is PR due to Proposition A.2 and above remark. However, they do not imply that $G(s)$ is PR in general as illustrated by the following example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The transfer function $G(s) = 1 - s$ is strictly positive real according to the above definition since all conditions are fulfilled. However, $G(s)$ is not positive real as $G(s) + G^H(s) = 2 - 2\text{Re}(s)$.

4.1. Kalman–Yakubovich–Popov lemma

When $E = I$, the following classical theorem summarizes well-known relationship between passivity of a system and positive realness of its transfer matrix.

Theorem 4.5. Consider the system (1) with $E = I$ and $m = p$. For the statements given below:

- (1) the system $\Sigma(I, A, B, C, D)$ is passive with a quadratic storage function;
- (2) the linear matrix inequalities

$$K = K^T \geq 0 \quad (14a)$$

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leq 0 \quad (14b)$$

have a solution K ;

- (3) the transfer matrix $D + C(sI - A)^{-1}B$ is positive real;
- (4) the quadruple (I, A, B, C) is minimal;

the following implications hold:

- (A) $1 \Leftrightarrow 2$.
- (B) $2 \Rightarrow 3$.
- (C) $3 \text{ and } 4 \Rightarrow 2$.

In a similar fashion, the following theorem summarizes the characterization of the extended strictly positive realness of a transfer matrix in terms of LMI for $E = I$.

Theorem 4.6. Consider the system (1) with $E = I$ and $m = p$. For the statements given below:

- (1) the linear matrix inequalities

$$K = K^T > 0 \quad (15a)$$

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} < 0 \quad (15b)$$

have a solution K ;

- (2) the transfer matrix $D + C(sI - A)^{-1}B$ is extended strictly positive real;
- (3) the quadruple (I, A, B, C) is minimal;
- (4) the matrix A is Hurwitz;

the following implications hold:

- (A) $1 \Rightarrow 2$;
- (B) $2 \text{ and } 3 \Rightarrow 2 \text{ and } 4 \Rightarrow 1$.

5. Main results

The first contribution of the paper is the following complete analogue of KYP lemma for descriptor systems.

Theorem 5.1. Consider the system (1) with $m = p$. Let \mathcal{W} be as in (13). For the statements given below:

- (1) the system $\Sigma(E, A, B, C, D)$ is passive with a quadratic storage function;
- (2) the linear matrix inequalities

$$K = K^T \geq 0 \quad (16a)$$

$$\begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \stackrel{\mathcal{W}}{\leq} 0 \quad (16b)$$

have a solution K ;

- (3) the transfer matrix $D + C(sE - A)^{-1}B$ is positive real;
- (4) the quadruple (E, A, B, C) is minimal;
- (5) the inclusion $\ker E \subseteq \ker K$ holds;

the following implications hold:

- (A) $1 \Leftrightarrow 2$;
- (B) $2 \Rightarrow 3$;
- (C) $3 \text{ and } 4 \Rightarrow 2$;
- (D) $2 \text{ and } 4 \Rightarrow 5$.

The second contribution is the analogue of Theorem 4.6 for passive descriptor systems.

Theorem 5.2. Consider the system (1) with $m = p$. Let \mathcal{W} and $\overline{\mathcal{W}}$ be as in (13). For the statements given below:

- (1) the linear matrix inequalities

$$K = K^T \geq 0 \quad (17a)$$

$$\begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \stackrel{\mathcal{W}}{\leq} 0$$

$$\begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \stackrel{\overline{\mathcal{W}}}{<} 0 \quad (17b)$$

have a solution K ;

- (2) the transfer matrix $D + C(sE - A)^{-1}B$ is extended strictly positive real;
- (3) the quadruple (E, A, B, C) is minimal;
- (4) the inclusion $\ker E \subseteq \ker K$ holds;

the following implications hold:

- (A) $1 \Rightarrow 2$.
- (B) $2 \text{ and } 3 \Rightarrow 1$;
- (C) $1 \text{ and } 3 \Rightarrow 4$.

Remark 5.3. The use of the subspaces \mathcal{W} and $\overline{\mathcal{W}}$ has two advantages. To begin with, it enables one to present necessary and sufficient LMI conditions as stated above. Furthermore, these LMI conditions look for symmetric solutions unlike typical LMI tests for passivity of descriptor systems (see for instance [3]). From the computational point of view, symmetric solutions are certainly preferable.

Before proving these theorems, we give a parametrization of all solution of the LMIs (16) and (17).

Theorem 5.4. Consider the descriptor system (1) with $m = p$. Suppose that

- (1) $D + C(sE - A)^{-1}B$ is positive real;
- (2) (E, A, B, C) is minimal;
- (3) (E, A, B, C) is given in the Weierstrass form of (A.1).

Then, the LMIs (16) are solvable and all solutions can be given by

$$K = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \quad (18)$$

where $K_{ii} \in \mathbb{R}^{n_i \times n_i}$ such that

(A) K_{11} is a solution of the LMIs (14) for

$$(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3), \text{ and}$$

(B) K_{33} is the unique solution of $B_3^T K_{33} = -C_2$.

Moreover, if $D + C(sE - A)^{-1}B$ is extended strictly positive real, then the LMIs (17) are solvable and all solutions can be given by Eq. (18) where

(A') K_{11} is a solution of the LMIs (15) for

$$(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3), \text{ and}$$

(B') K_{33} is the unique solution of $B_3^T K_{33} = -C_2$.

Proof. From the hypotheses 3, we have

$$(E, A, B, C) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \right). \quad (19)$$

Then, straightforward calculations yield

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ 0 & 0 & -B_2 & -B_3 \\ 0 & 0 & -B_3 & 0 \\ I & 0 & 0 & 0 \\ 0 & -B_2 & -B_3 & 0 \\ 0 & -B_3 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix} W = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & -B_2 \\ 0 & -B_3 \\ 0 & I \end{bmatrix}.$$

Let K be a symmetric positive semi-definite matrix and be partitioned as

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12}^T & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{bmatrix} = K^T \geq 0 \quad (20)$$

where $K_{ij} \in \mathbb{R}^{n_i \times n_j}$. Then, it can be verified that the following equations hold

$$\begin{aligned} W^T \begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} W \\ = \begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) \end{bmatrix}, \end{aligned} \quad (21)$$

and

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} = \begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & \mathcal{M}_{12} & \mathcal{M}_{13} & -K_{12} B_3 \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} \\ \mathcal{M}_{13}^T & \mathcal{M}_{23}^T & \mathcal{M}_{33} & B_3^T K_{22} B_3 \\ -B_3^T K_{12}^T & \mathcal{M}_{24}^T & B_3^T K_{22} B_3 & \mathcal{M}_{44} \end{bmatrix}, \quad (22)$$

where

$$\begin{aligned} \mathcal{M}_{12} &= K_{11} B_1 - A_1^T K_{12} B_2 - A_1^T K_{13} B_3 - C_1^T \\ \mathcal{M}_{13} &= -K_{12} B_2 - K_{13} B_3 - A_1^T K_{12} B_3 \\ \mathcal{M}_{22} &= -B_2^T K_{12}^T B_1 - B_3^T K_{13}^T B_1 - B_1^T K_{12} B_2 - B_1^T K_{13} B_3 + C_2 B_2 \\ &\quad + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) \\ \mathcal{M}_{23} &= B_2^T K_{22} B_2 + B_3^T K_{23}^T B_2 + B_2^T K_{23} B_3 + B_3^T K_{33} B_3 - B_1^T K_{12} B_3 + C_2 B_2 \\ \mathcal{M}_{33} &= B_3^T K_{22} B_2 + B_3^T K_{23} B_3 + B_2^T K_{22} B_3 + B_3^T K_{23}^T B_3 \\ \mathcal{M}_{24} &= B_2^T K_{22} B_3 + B_3^T K_{23}^T B_3 \\ \mathcal{M}_{44} &= 0. \end{aligned}$$

The rest of the proof follows from the following auxiliary lemmas. ■

Lemma 5.5. The right-hand side of (22) is negative semi-definite if and only if

- (1) K_{12}, K_{13}, K_{22} and K_{23} are zero matrices;
- (2) K_{11} is a solution to LMIs (14) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$;
- (3) K_{33} is the unique solution of $B_3^T K_{33} = -C_2$.

Proof. 'if': Straightforward calculations show that the right-hand side of (22) boils down to

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T & 0 & 0 \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (23)$$

Since K_{11} is a solution to LMIs (14) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$, this matrix is negative semi-definite.

'only if': Suppose that (22) is negative semi-definite. Since $\mathcal{M}_{44} = 0$, it follows from elementary linear algebra that all the elements on the corresponding row and column blocks must be zero. In other words,

$$B_3^T K_{22} B_3 = 0 \quad (24a)$$

$$\mathcal{M}_{24} = 0 \quad (24b)$$

$$-K_{12} B_3 = 0. \quad (24c)$$

Due to minimality, B_3 is of full row rank. Then, (24) yields that $K_{12} = 0$, $K_{22} = 0$ and $K_{23} = 0$. As such, we get $\mathcal{M}_{33} = 0$. Hence, both \mathcal{M}_{13} and \mathcal{M}_{23} must be zero due to definiteness. This yields $K_{13} = 0$ and $B_3^T K_{33} + C_2 = 0$ as B_3 is of full row rank. Hence, K_{33} is the unique solution of $B_3^T K_{33} = -C_2$. Therefore, the right-hand side of (22) boils down to

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T & 0 & 0 \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leq 0. \quad (25)$$

Hence, K_{11} is a solution to LMIs (14) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$. ■

Lemma 5.6. *The right-hand side of (22) is negative semi-definite and the right-hand side of (21) is negative definite if and only if*

- (1) K_{12}, K_{13}, K_{22} and K_{23} are zero matrices;
- (2) K_{11} is a solution to LMIs (15) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$;
- (3) K_{33} is the unique solution of $B_3^T K_{33} = -C_2$.

Proof. ‘if’: Straightforward calculations show that the right-hand side of (22) boils down to

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T & 0 & 0 \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

and that of (21) to

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) \end{bmatrix}. \quad (27)$$

Since K_{11} is a solution to LMIs (15) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$, the latter matrix is negative definite whereas the former is negative semi-definite.

‘only if’: Suppose that (22) is negative semi-definite. By following the same steps as in Lemma 5.5, it is simple to see that conditions 1 and 3 hold. Since (21) is negative definite by hypothesis, we get

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D + D^T) \end{bmatrix} < 0. \quad (28)$$

Hence, K_{11} is a solution to LMIs (15) with $(A, B, C, D) = (A_1, B_1, C_1, D - C_2B_2 - C_3B_3)$. ■

In what follows, we prove the main contributions of the paper.

5.1. Proof of Theorem 5.1

A: ‘1 \Rightarrow 2’ Suppose that the system $\Sigma(E, A, B, C, D)$ is passive with the quadratic storage function

$$V(x) = \frac{1}{2} x^T K x \quad (29)$$

where $K \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Let $(x, u) \in \mathcal{B}_s$ and $y = Cx + Du$. From the dissipation inequality, we get

$$\int_{t_0}^{t_1} u^T(t)y(t) dt \geq \int_{t_0}^{t_1} \frac{dV}{dt}(t) dt \quad (30)$$

for all $t_1 \geq t_0$. Since all involved functions are smooth, we get

$$u^T(t)y(t) \geq \frac{dV}{dt}(t) \quad (31)$$

for all $t \in \mathbb{R}$ by differentiation. By using (1) and (29), we get

$$\begin{bmatrix} \dot{x}(0) \\ x(0) \\ u(0) \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} \dot{x}(0) \\ x(0) \\ u(0) \end{bmatrix} \leq 0. \quad (32)$$

It follows from Lemma 3.2.2 that K is a solution to the LMIs (16).

A: ‘2 \Rightarrow 1’ Suppose that the LMIs (16) admit a solution K . From Lemma 3.2.1, we get that

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \\ u(t) \end{bmatrix} \leq 0 \quad (33)$$

for almost all $t \in \mathbb{R}$ whenever $(x, u) \in \mathcal{B}$. Clearly, this yields

$$u^T(t)y(t) \geq \frac{1}{2} \frac{d}{dt} (x^T(t)Kx(t)) \quad (34)$$

for almost all $t \in \mathbb{R}$. By integrating from t_0 to t_1 , we get the dissipation inequality.

B: Let $s \in \mathbb{C}_+$ be any point such that s is not a pole of $G(s)$. This means that $sE - A$ and $sI - F$ are both invertible. Let V, F, U be as in Lemma 3.2. Let $\xi \in \ker V \cap \mathbb{C}^\ell$. Define $X(s) = V(sI - F)^{-1}\xi$, $U(s) = U(sI - F)^{-1}\xi$ and $w = \text{col}(sX(s), X(s), U(s))$. Since $\xi \in \ker V$, we get

$$\begin{aligned} sX(s) &= sV(sI - F)^{-1}\xi - V\xi \\ &= V[s(sI - F)^{-1} - I]\xi = VF(sI - F)^{-1}\xi. \end{aligned}$$

Thus $w \in \mathcal{W}$. Then

$$sEX(s) = AX(s) + BU(s). \quad (35)$$

Note that

$$\begin{aligned} 0 &\geq w^H \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} w \\ &= \text{Re}(s)X^H(s)KX(s) - X^H(s)C^T U(s) \\ &\quad - U^H(s)CX(s) - U^H(s)(D + D^T)U(s). \end{aligned} \quad (36)$$

Since $s \in \mathbb{C}_+$ and K is positive semi-definite, (36) results in

$$U^H(s)[CX(s) + DU(s)] + [CX(s) + DU(s)]^H U(s) \geq 0.$$

By solving $X(s)$ from (35), we get

$$U^H(s)[G(s) + G^H(s)]U(s) \geq 0. \quad (37)$$

To conclude the proof, we need to show that $U(sI - F)^{-1}(\ker V \cap \mathbb{C}^\ell) = \mathbb{C}^m$. This can be achieved by assuming that (1) is given in the Weierstrass form and using (10) and (11). Now, suppose that $G(s)$ has a pole $s_0 \in \mathbb{C}_+$. This means that the condition (37) holds in a pointed neighborhood of s_0 which is free of any pole. This, however, would contradict to the fact that s_0 is a pole. Thus, $G(s)$ does not have any pole in \mathbb{C}_+ and (37) holds for all $s \in \mathbb{C}_+$. So, $G(s)$ is positive real.

C: In view of Proposition A.4, we can assume without loss of generality that (E, A, B, C) is given in the form (19). Then, the proof follows from Theorem 5.4.

D: Note that $G(s)$ is positive real, due to statement B. The rest follows from Proposition A.4 and Theorem 5.4. ■

5.2. Proof of Theorem 5.2

A: By using Theorem 5.1, (17a) and (17b), we can conclude that $G(s)$ is positive real. Now, consider $s_0 \in \mathbb{C}$ such that $|s_0 E - A| = 0$. This means that $s_0 E x_0 = A x_0$ for some $x_0 \in \mathbb{C}$. Since $x(t) = e^{s_0 t} x_0$ is a solution of $E \dot{x} = Ax$, we get that $\dot{x}(t) = s_0 x(t)$. This implies that $\text{col}(s_0 x_0, x_0, 0) \in \overline{\mathcal{W}}$. From (17b), we get

$$0 > \begin{bmatrix} s_0 x_0 \\ x_0 \end{bmatrix}^H \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \begin{bmatrix} s_0 x_0 \\ x_0 \end{bmatrix} = 2\text{Re}(s_0) x_0^H K x_0. \quad (38)$$

Since $K \geq 0$, this means that $\text{Re}(s_0) < 0$ and thus $G(s)$ is analytic in $\text{Re}(s_0) \geq 0$.

Without loss of generality, we can assume that (E, A, B, C) is in the Weierstrass form (3). Then, it follows from the proof of Lemma 3.2 that

$$\mathcal{W} = \text{im} \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \quad \text{and} \quad \overline{\mathcal{W}} = \text{im} Z_1 \quad (39)$$

where $Z_1 \in \mathbb{R}^{[n_1+(k+1)m] \times (n_1+m)}$ and $Z_2 \in \mathbb{R}^{[n_1+(k+1)m] \times km}$ are given by

$$Z_1 = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \\ I & 0 \\ 0 & -B_2 \\ 0 & I \end{bmatrix}, \quad (40)$$

$$Z_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -B_2 & -NB_2 & \cdots & -N^{k-2}B_2 & -N^{k-1}B_2 \\ 0 & 0 & \cdots & 0 & 0 \\ -NB_2 & -N^2B_2 & \cdots & -N^{k-1}B_2 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Let

$$Q = \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix}.$$

Then, (17b) can be written as

$$\begin{bmatrix} Z_1^T Q Z_1 & Z_1^T Q Z_2 \\ Z_2^T Q Z_1 & Z_2^T Q Z_2 \end{bmatrix} \leq 0 \quad (41)$$

$$Z_1^T Q Z_1 < 0. \quad (42)$$

We claim that $Z_2^T Q Z_2 = 0$. To see this, note that

$$Z_2^T Q Z_2 = \text{Sym} \left\{ \begin{bmatrix} B_2 & NB_2 & \cdots & N^{k-2}B_2 & N^{k-1}B_2 \end{bmatrix}^T \right. \\ \left. \times K_{22} \begin{bmatrix} NB_2 & N^2B_2 & \cdots & N^{k-1}B_2 & N^kB_2 \end{bmatrix} \right\} \quad (43)$$

where K is partitioned accordingly to the Weierstrass form as

$$K = K^T = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}. \quad (44)$$

Let $(Z_2^T Q Z_2)_{ij}$ denote the submatrix of $Z_2^T Q Z_2$ by taking the rows $(i-1)m+1, (i-1)m+2, \dots, im$ and the columns $(j-1)m+1, (j-1)m+2, \dots, jm$, where $i, j \in \{1, 2, \dots, k\}$. Then, we get

$$(Z_2^T Q Z_2)_{ij} = B_2^T (N^{i-1})^T K_{22} N^j B_2 + B_2^T (N^{j-1})^T K_{22} N^i B_2 \quad (45)$$

from (43) by noting that $N^k B_2 = 0$. Observe that $(Z_2^T Q Z_2)_{kk} = 0$. Since $Z_2^T Q Z_2$ is negative semi-definite due to (41), we get $(Z_2^T Q Z_2)_{ik} = 0$ for all $i = 1, 2, \dots, k$. Hence, we get

$$0 = B_2^T (N^{i-1})^T K_{22} N^k B_2 + B_2^T (N^{k-1})^T K_{22} N^i B_2 \\ = B_2^T (N^{k-1})^T K_{22} N^i B_2 \quad (46)$$

Note that

$$(Z_2^T Q Z_2)_{(k-1)(k-1)} \\ = B_2^T (N^{k-2})^T K_{22} N^{k-1} B_2 + B_2^T (N^{k-1})^T K_{22} N^{k-2} B_2 = 0. \quad (47)$$

By repeating the above steps, we get

$$(Z_2^T Q Z_2)_{ii} = 0 \quad (48)$$

for all i . The negative semi-definiteness of $Z_2^T Q Z_2$ and (48) allow us to conclude that $Z_2^T Q Z_2 = 0$. Moreover, this implies that also $Z_1^T Q Z_2 = 0$. Hence, we get

$$K_{12} N^{i-1} B_2 + A_1^T K_{12} N^i B_2 = 0 \\ \text{for all } i \in \{1, 2, \dots, k-1\} \quad (49)$$

$$K_{12} N^{k-1} B_2 = 0 \quad (50)$$

$$-B_1^T K_{12} N^i B_2 + B_2^T K_{22} N^{i-1} B_2 + C_2 N^i B_2 = 0 \\ \text{for all } i \in \{1, 2, \dots, k-1\} \quad (51)$$

$$B_2^T K_{22} N^{k-1} B_2 = 0. \quad (52)$$

Thus, from Eqs. (49) and (50), we can conclude that

$$K_{12} N^i B_2 = 0 \quad \text{for all } i \in \{0, 1, \dots, k-1\}. \quad (53)$$

Since by hypotheses $G(s)$ is positive real, it follows from Proposition A.2 that

$$C_2 N^\ell B_2 = 0 \quad (54)$$

for all $\ell = \{2, 3, \dots, k-1\}$. Thus, by using Eqs. (51)–(53), we get

$$B_2^T K_{22} B_2 + C_2 N B_2 = 0. \quad (55)$$

Note that (42) boils down to

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & -A_1^T K_{12} B_2 + K_{11} B_1 - C_1^T \\ -B_2^T K_{12}^T A_1 + B_1^T K_{11} - C_1 & -B_2^T K_{12}^T B_1 + C_2 B_2 + B_2^T C_2^T - (D + D^T) \end{bmatrix} \\ < 0. \quad (56)$$

Since $K_{12} B_2 = 0$, we can conclude that $G_1(s) = C_1(sI - A_1^{-1})B_1 + D - C_2 B_2$ is ESPR by applying Theorem 4.6. Note that $G(s) = C_1(sI - A_1^{-1})B_1 + D - C_2 B_2 - \sum_{\ell=1}^{k-1} C_2 N^\ell B_2 s^\ell$. It follows from (54) and (55) that $G(s) = G_1(s) + B_2^T K_{22} B_2 s$. Since $G_1(s)$ is ESPR and $B_2^T K_{22} B_2$ is positive semi-definite, it follows from Proposition A.2 that $G(s)$ is ESPR.

B: In view of Proposition A.4, we can assume without loss of generality that (E, A, B, C) is given in the form (19). Then, the proof follows from Theorem 5.4

C: Note that $G(s)$ is extended strictly positive real, due to statement A. The rest follows from Proposition A.4 and Theorem 5.4. ■

6. Comparison with previous results

In what follows, we will compare our results with the available results in the literature.

Remark 6.1 (KYP Lemma). When $E = I$, it can be verified that

$$\mathcal{W} = \overline{\mathcal{W}} = \ker \begin{bmatrix} E & -A & -B \end{bmatrix} = \text{im} \begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix}.$$

Since

$$\begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix} \\ = \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C^T & -(D + D^T) \end{bmatrix},$$

Kalman–Yakubovich–Popov lemma is recovered as a special case from Theorem 5.1.

Remark 6.2 ([13]). In [13, Corollary 4.1], it is claimed that a minimal descriptor system (1) with $m = p$ is passive if, and only if, the LMIs

$$P = P^T > 0 \quad \begin{bmatrix} A^T P E + E^T P A & P B - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \leq 0 \quad (57)$$

admit a solution. However, this result cannot hold in this generality as $D + D^T$ is not necessarily positive semi-definite for the passive descriptor system. As an example, consider the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - u.$$

Since $D = -1$, the LMIs (57) do not admit a solution. However, the dissipation inequality holds for the storage function $V(x) = \frac{1}{2}x_2^2$.

Remark 6.3. The following theorem summarizes the extension of Kalman–Yakubovich–Popov lemma to the descriptor system that is proposed in [3].

Theorem 6.4 ([3, Theorem 1, 2 and 3]). Consider the descriptor system (1) with $m = p$. Let $G(s) = C(sE - A)^{-1}B + D$. The following statement hold.

(1) If the LMIs

$$E^T X = X^T E \geq 0 \quad (58a)$$

$$\begin{bmatrix} A^T X + X^T A & X^T B - C^T \\ B^T X - C & -(D + D^T) \end{bmatrix} \leq 0 \quad (58b)$$

admit a solution, then $G(s)$ is positive real.

(2) If the LMIs

$$E^T X E = E^T X^T E \geq 0 \quad (59a)$$

$$\begin{bmatrix} A^T X E + E^T X^T A & E^T X^T B - C^T \\ B^T X E - C & -(D + D^T) \end{bmatrix} \leq 0 \quad (59b)$$

admit a solution, then $G(s)$ is positive real.

(3) Suppose that $G(s)$ is positive real and $G(s) = G_1(s) + sG_0$ where $G_1(s)$ is proper. If (E, A, B, C) is minimal and $D + D^T \geq G_1(\infty) + G_1^T(\infty)$ then the LMIs (58) admit a solution.

By taking $K = E^T X$ and using $EVF = AV + BU$, we get

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}$$

$$= \begin{bmatrix} V \\ U \end{bmatrix}^T \begin{bmatrix} A^T X + X^T A & X^T B - C^T \\ B^T X - C & -(D + D^T) \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}.$$

Similarly, by taking $K = E^T X E$ and using $EVF = AV + BU$, we get

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D + D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}$$

$$= \begin{bmatrix} V \\ U \end{bmatrix}^T \begin{bmatrix} A^T X E + E^T X^T A & E^T X^T B - C^T \\ B^T X E - C & -(D + D^T) \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}.$$

It follows from the last two equations that the LMIs (16) admit a solution whenever one of the LMIs (58) and (59) admits a solution.

To see that the last statement follows from our main result, assume that (E, A, B, C) is given in the Weierstrass form of (A.1).

Note that $G_1(\infty) = D - C_2 B_2 - C_3 B_3$ and also that

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & -(D + D^T) \end{bmatrix}$$

$$\leq \begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & -(G_1(\infty) + G_1^T(\infty)) \end{bmatrix}$$

as $D + D^T \geq G_1(\infty) + G_1^T(\infty)$. Then, it follows from Theorem 5.4 that the LMIs (58) admits a solution with

$$X = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & K_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

7. Conclusions

This paper presents a complete analogue of Kalman–Yakubovich–Popov lemma for regular descriptor systems. Unlike previous work, we do not make any additional assumptions such as impulse-freeness or impose any condition on the feed-through term. We also give a full characterization of the so-called extended strict positive realness for descriptor systems. After establishing our main results, some of the earlier results are recovered as special cases.

Appendix. Auxiliary results

A characterization of positive realness can be found in the following theorem:

Theorem A.1 ([14, Theorem 2.7.2]). A real rational function $G : \mathbb{C} \mapsto (\mathbb{C} \cup \infty)^{m \times m}$ is positive real if, and only if, the following conditions are satisfied:

- G has no poles in \mathbb{C}_+ ;
- $G(i\omega) + G^H(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $i\omega$ not a pole of G ;
- If $i\omega$ or ∞ is a pole of G , then it is a simple pole and the associated residue matrix is positive semi-definite.

Proposition A.2. Let (E, A, B, C) be given such that $G(s) = D + C(sE - A)^{-1}B$ is the transfer matrix. Then, $G(s)$ is PR/SPR/ESPR if and only if $G(s) = G_1(s) + G_0 s$ where G_1 is proper and PR/SPR/ESPR and $G_0 = G_0^T \geq 0$.

Proof. This follows from Theorem A.1. by noting that $G(i\omega) + G^H(i\omega) = G_1(i\omega) + G_1^H(i\omega)$. ■

The Weierstrass form plays a key role in the analysis of descriptor systems. The following proposition imposes a particular structure on the Weierstrass form of the systems of interests in this paper.

Proposition A.3. Let (E, A, B, C, D) be given such that

- (E, A, B, C) is minimal, and
- if $s = \infty$ is a pole of $D + C(sE - A)^{-1}B$ then it is a simple pole.

Then, there exist matrices $(S, T) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ such that

$$(SET, SAT, SB, CT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, [C_1 \ C_2 \ C_3] \right) \quad (A.1)$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $B_i \in \mathbb{R}^{n_i \times m}$, $C_i \in \mathbb{R}^{n_i \times m}$ and all other matrices involved are of appropriate sizes.

Proof. The same result is obtained by [3, Proposition 2] where the second condition is replaced by positive realness. This proposition still holds if one replaces positive realness by the second condition. This observation concludes our proof. ■

Proposition A.4. Let (E, A, B, C, D) be given such that (E, A, B, C) is minimal and $C(sE - A)^{-1}B + D$ is positive real. Then, (E, A, B, C) admits the Weierstrass form (A.1). Moreover,

- (1) $C(sE - A)^{-1}B + D = C_1(sI - A_1)^{-1}B_1 + D - C_2B_2 - C_3B_3 - sC_2B_3$ and C_2B_3 is negative semi-definite;
- (2) (I, A_1, B_1, C_1) is minimal;
- (3) B_3 is of full row rank.

Proof. The statements follow from Theorems A.1, 3.1, and Propositions A.2 and A.3. ■

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